

A NOTE ON GEOMETRY OF κ -MINKOWSKI SPACE

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ABSTRACT. The infinitesimal action of κ -Poincaré group on κ -Minkowski space is computed both for generators of κ -Poincaré algebra and those of Woronowicz generalized Lie algebra. The notion of invariant operators is introduced and generalized Klein-Gordon equation is written out.

I. INTRODUCTION

In this short note we consider some simple properties of differential operators on κ -Minkowski space \mathcal{M}_κ — a noncommutative deformation of Minkowski space-time which depends on dimensionful parameter κ ([1]). We calculate the infinitesimal action of κ -Poincaré group \mathcal{P}_κ ([1]) on \mathcal{M}_κ both for the generators of κ -Poincaré algebra $\tilde{\mathcal{P}}_\kappa$ ([2]) (this is done using the duality $\tilde{\mathcal{P}}_\kappa \iff \mathcal{P}_\kappa$ described in [3]) and for the elements of Woronowicz generalized Lie algebra ([4]) of κ -Poincaré group ([5]). The result supports the relation between both algebras found in [5]. We introduce also the notion of invariant differential operators on \mathcal{M}_κ and write out the generalized Klein-Gordon equation.

Let us conclude this section by introducing the notions of κ -Poincaré group \mathcal{P}_κ and algebra $\tilde{\mathcal{P}}_\kappa$. \mathcal{P}_κ is defined by the following relations ([1])

$$\begin{aligned} [x^\mu, x^\nu] &= \frac{i}{\kappa}(\delta_0^\mu x^\nu - \delta_0^\nu x^\mu), \\ [\Lambda^\mu{}_\nu, \Lambda^\alpha{}_\beta] &= 0, \\ [\Lambda^\mu{}_\nu, x^\rho] &= -\frac{i}{\kappa}((\Lambda^\mu{}_0 - \delta_0^\mu)\Lambda^\rho{}_\nu + (\Lambda^0{}_\nu - \delta_\nu^0)g^{\mu\rho}), \\ \Delta(\Lambda^\mu{}_\nu) &= \Lambda^\mu{}_\alpha \otimes \Lambda^\alpha{}_\nu, \\ \Delta(x^\mu) &= \Lambda^\mu{}_\alpha \otimes x^\alpha + x^\mu \otimes I, \\ S(\Lambda^\mu{}_\nu) &= \Lambda_\nu{}^\mu, \\ S(x^\mu) &= -\Lambda_\nu{}^\mu x^\nu, \\ \varepsilon(\Lambda^\mu{}_\nu) &= \delta_\nu^\mu, \\ \varepsilon(x^\mu) &= 0. \end{aligned} \tag{1}$$

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The dual structure, the κ -Poincaré algebra $\tilde{\mathcal{P}}_\kappa$, is, in turn, defined as follows ([6])

$$\begin{aligned}
[P_\mu, P_\nu] &= 0, \\
[M_i, M_j] &= i\varepsilon_{ijk}M_k, \\
[M_i, N_j] &= i\varepsilon_{ijk}N_k, \\
[N_i, N_j] &= -i\varepsilon_{ijk}M_k, \\
[M_i, P_0] &= 0, \\
[M_i, P_j] &= i\varepsilon_{ijk}P_k, \\
[N_i, P_0] &= iP_i, \\
[N_i, P_j] &= i\delta_{ij}\left(\frac{\kappa}{2}(1-e^{-2P_0/\kappa}) + \frac{1}{2\kappa}\vec{P}^2\right) - \frac{i}{\kappa}P_iP_j, \\
\Delta(M_i) &= M_i \otimes I + I \otimes M_i, \\
\Delta(N_i) &= N_i \otimes e^{-P_0/\kappa} + I \otimes N_i - \frac{1}{\kappa}\varepsilon_{ijk}M_j \otimes P_k, \\
\Delta(P_0) &= P_0 \otimes I + I \otimes P_0, \\
\Delta(P_i) &= P_i \otimes e^{-P_0/\kappa} + I \otimes P_i, \\
S(M_i) &= -M_i, \\
S(N_i) &= -N_i + \frac{3i}{2\kappa}P_i, \\
S(P_\mu) &= -P_\mu, \\
\varepsilon(P_\mu, M_i, N_i) &= 0.
\end{aligned} \tag{2}$$

Structures (1), (2) are dual to each other, the duality being fully described in [3].

The analysis given below was suggested to two of the authors (P. Kosiński and P. Maślanka) by J. Lukierski.

II. κ -MINKOWSKI SPACE

The κ -Minkowski space \mathcal{M}_κ ([1]) is a universal $*$ -algebra with unity generated by four selfadjoint elements x^μ subject to the following conditions

$$[x^\mu, x^\nu] = \frac{i}{\kappa}(\delta_0^\mu x^\nu - \delta_0^\nu x^\mu). \tag{3a}$$

Equipped with the standard coproduct

$$\Delta x^\mu = x^\mu \otimes I + I \otimes x^\mu, \tag{3b}$$

antipode $S(x^\mu) = -x^\mu$ and counit $\varepsilon(x^\mu) = 0$ it becomes a quantum group.

On \mathcal{M}_κ one can construct a bicovariant five-dimensional calculus which is defined

by the following relations ([5])

$$\begin{aligned}
\tau^\mu &\equiv dx^\mu, & \tau &\equiv d\left(x^2 + \frac{3i}{\kappa}x^0\right) - 2x_\mu dx^\mu, \\
[\tau^\mu, x^\nu] &= \frac{i}{\kappa}g^{0\mu}\tau^\nu - \frac{i}{\kappa}g^{\mu\nu}\tau^0 + \frac{1}{4}g^{\mu\nu}\tau, \\
[\tau, x^\mu] &= -\frac{4}{\kappa^2}\tau^\mu, \\
\tau^\mu \wedge \tau^\nu &= -\tau^\nu \wedge \tau^\mu, \\
\tau \wedge \tau^\mu &= -\tau^\mu \wedge \tau, \\
(\tau^\mu)^* &= \tau^\mu, & \tau^* &= -\tau, \\
d\tau^\mu &= 0, \\
d\tau &= -2\tau_\mu \wedge \tau^\mu.
\end{aligned} \tag{4}$$

The κ -Minkowski space carries a left-covariant action of κ -Poincaré group \mathcal{P}_κ ([1]), $\rho_L : \mathcal{M}_\kappa \rightarrow \mathcal{P}_\kappa \otimes \mathcal{M}_\kappa$, given by

$$\rho_L(x^\mu) = \Lambda^\mu{}_\nu \otimes x^\nu + a^\mu \otimes I. \tag{5}$$

The calculus defined by (4) is covariant under the action of \mathcal{P}_κ which reads

$$\begin{aligned}
\tilde{\rho}_L(\tau^\mu) &= \Lambda^\mu{}_\nu \otimes \tau^\nu, \\
\tilde{\rho}_L(\tau) &= I \otimes \tau.
\end{aligned} \tag{6}$$

III. DERIVATIVES, INFINITESIMAL ACTIONS AND INVARIANT OPERATORS

The product of generators x^μ will be called normally ordered if all x^0 factors stand leftmost. This definition can be used to ascribe a unique element : $f(x)$: of \mathcal{M}_κ to any polynomial function of four variables f . Formally, it can be extended to any analytic function f .

Let us now one define the (left) partial derivatives: for any $f \in \mathcal{M}_\kappa$ we write

$$df = \partial_\mu f \tau^\mu + \partial f \tau. \tag{7}$$

It is a matter of some boring calculations (using the commutation rules (3a)) to find the following formula

$$\begin{aligned}
d : f :=: & \left(\kappa \sin\left(\frac{\partial_0}{\kappa}\right) + \frac{i}{2\kappa} e^{i\frac{\partial_0}{\kappa}} \Delta \right) f : \tau^0 + : e^{i\frac{\partial_0}{\kappa}} \frac{\partial f}{\partial x^i} : \tau^i \\
& + : \left(\frac{\kappa^2}{4} \left(1 - \cos\left(\frac{\partial_0}{\kappa}\right) \right) - \frac{1}{8} e^{i\frac{\partial_0}{\kappa}} \Delta \right) f : \tau
\end{aligned} \tag{8}$$

or

$$\begin{aligned}
\partial_0 : f :=: & \left(\kappa \sin\left(\frac{\partial_0}{\kappa}\right) + \frac{i}{2\kappa} e^{i\frac{\partial_0}{\kappa}} \Delta \right) f : \\
\partial_i : f :=: & e^{i\frac{\partial_0}{\kappa}} \frac{\partial f}{\partial x^i} : \\
\partial : f :=: & \left(\frac{\kappa^2}{4} \left(1 - \cos\left(\frac{\partial_0}{\kappa}\right) \right) - \frac{1}{8} e^{i\frac{\partial_0}{\kappa}} \Delta \right) f :
\end{aligned} \tag{9}$$

Let us now define the infinitesimal action of \mathcal{P}_κ on \mathcal{M}_κ . Let X be any element of the Hopf algebra dual to \mathcal{P}_κ — the κ -Poincaré algebra $\tilde{\mathcal{P}}_\kappa$ (cf. [3] for the proof of duality). The corresponding infinitesimal action

$$\hat{X} : \mathcal{M}_\kappa \rightarrow \mathcal{M}_\kappa$$

is defined as follows: for any $f \in \mathcal{M}_\kappa$,

$$\hat{X}f = (X \otimes \text{id}) \circ \rho_L(f). \quad (10)$$

Using the standard duality rules ([3]), we conclude that

$$\begin{aligned} \hat{P}_\mu x^\alpha &= i\delta_\mu^\alpha, \\ \hat{P}_\mu : x^\alpha x^\beta &:= i\delta_\mu^\beta x^\alpha + i\delta_\mu^\alpha x^\beta \end{aligned} \quad (11)$$

etc. One can show that, in general,

$$\hat{P}_\mu : f :=: i \frac{\partial f}{\partial x^\mu} : \quad (12)$$

Also, using the fact that $\tilde{\rho}_L$ is a left action of \mathcal{P}_κ on \mathcal{M}_κ together with the duality $\mathcal{P}_\kappa \rightarrow \tilde{\mathcal{P}}_\kappa$, we conclude that

$$F(\hat{P}_\mu) : f :=: F\left(i \frac{\partial f}{\partial x^\mu}\right) f : \quad (13)$$

Formulae (11)–(13) have the following interpretation. In [5] the fifteen-dimensional bicovariant calculus on \mathcal{P}_κ has been constructed using the methods developed by Woronowicz ([4]). The resulting generalized Lie algebra is also fifteen-dimensional, the additional generators being the generalized mass square operator and the components of generalized Pauli-Lubanski fourvector. All generators of this Lie algebra can be expressed in terms of the generators P_μ , $M_{\alpha\beta}$ of $\tilde{\mathcal{P}}_\kappa$ ([5]). In particular, the translation generators χ_μ as well as the mass squared operator χ are expressible in terms of P_μ only. The relevant expressions are given by formulae (20) of [5]. Comparing them with (9), (13) above, we conclude that

$$\begin{aligned} \hat{\chi}_\mu &\equiv \partial_\mu, \\ \hat{\chi} &\equiv \partial. \end{aligned} \quad (14)$$

These relations, obtained here by explicit computations, follow also from (7) if one takes into account that \mathcal{M}_κ is a quantum subgroup of \mathcal{P}_κ .

It is also not difficult to obtain the action of Lorentz generators. Combining (1) and (3a) with the duality $\mathcal{P}_\kappa \rightarrow \tilde{\mathcal{P}}_\kappa$ described in detail in [5], we conclude first that the action of M_i and N_i coincides with the proposal of Majid and Ruegg ([6]); the actual computation is then easy and gives

$$\begin{aligned} \widehat{M}_i : f(x^\mu) &=: -i\varepsilon_{ijl}x^j \frac{\partial f(x^\mu)}{\partial x^l} : \\ \widehat{N}_i : f(x^\mu) &=: \left(ix^0 \frac{\partial}{\partial x^i} + x^i \left(\frac{\kappa}{2} \left(1 - e^{-\frac{2i}{\kappa} \frac{\partial}{\partial x^0}} \right) - \frac{1}{2\kappa} \Delta \right) \right. \\ &\quad \left. + \frac{1}{\kappa} x^k \frac{\partial^2}{\partial x^k \partial x^i} \right) f(x^\mu) : \end{aligned} \quad (15)$$

Let us now pass to the notion of invariant operator; \widehat{C} is an invariant operator on \mathcal{M}_κ if

$$\rho_L \circ \widehat{C} = (\text{id} \otimes \widehat{C}) \circ \rho_L. \quad (16)$$

We shall show that if C is a central element of $\widetilde{\mathcal{P}}_\kappa$, then

$$\widehat{C}f = (C \otimes \text{id}) \circ \rho_L(f) \quad (17)$$

is an invariant operator. To prove this let us take any $Y \in \widetilde{\mathcal{P}}_\kappa$, then

$$YC = CY \quad (18)$$

or, in other words, for any $a \in \mathcal{P}_\kappa$,

$$Y(a_{(1)})C(a_{(2)}) = C(a_{(1)})Y(a_{(2)}) \quad (19)$$

where $\Delta a = a_{(1)} \otimes a_{(2)}$. Let us fix a and write (19) as

$$Y(a_{(1)}C(a_{(2)})) = Y(a_{(2)}C(a_{(1)})). \quad (20)$$

As (20) holds for any $Y \in \widetilde{\mathcal{P}}_\kappa$ we conclude that for any $a \in \mathcal{P}_\kappa$

$$C(a_{(2)})a_{(1)} = C(a_{(1)})a_{(2)}. \quad (21)$$

Now let

$$\begin{aligned} \rho_L(x) &= a_{(1)} \otimes x_{(1)}, \\ \rho_L(x_{(1)}) &= a_{(1)(2)} \otimes x_{(2)}, \\ \Delta a_{(1)} &= a_{(1)}^{(1)} \otimes a_{(1)}^{(2)}. \end{aligned} \quad (22)$$

The identity

$$(\text{id} \otimes \rho_L) \circ \rho_L = (\Delta \otimes \text{id}) \circ \rho_L \quad (23)$$

implies

$$a_{(1)} \otimes a_{(1)(2)} \otimes x_{(2)} = a_{(1)}^{(1)} \otimes a_{(1)}^{(2)} \otimes x_{(1)}. \quad (24)$$

Applying to both sides $\text{id} \otimes C \otimes \text{id}$ and $C \otimes \text{id} \otimes \text{id}$, we get

$$\begin{aligned} C(a_{(1)(2)})a_{(1)} \otimes x_{(2)} &= C(a_{(1)}^{(2)})a_{(1)}^{(1)} \otimes x_{(1)}, \\ C(a_{(1)})a_{(1)(2)} \otimes x_{(2)} &= C(a_{(1)}^{(1)})a_{(1)}^{(2)} \otimes x_{(1)}. \end{aligned} \quad (25)$$

It follows from (21) applied to $a_{(1)}$ that the right-hand sides of (25) are equal. So,

$$C(a_{(1)(2)})a_{(1)} \otimes x_{(2)} = C(a_{(1)})a_{(1)(2)} \otimes x_{(2)} \quad (26)$$

i.e.

$$(\text{id} \otimes \widehat{C}) \circ \rho_L(x) = \rho_L \circ \widehat{C}(x). \quad (27)$$

Using the above result we can easily construct the deformed Klein-Gordon equation. Namely, we take as a central element the counterpart of mass squared Casimir operator χ ([5]). Due to (14) the generalized Klein-Gordon equation reads

$$\left(\partial + \frac{m^2}{8} \right) f = 0; \quad (28)$$

the coefficient $\frac{1}{8}$ is dictated by the correspondence with standard Klein-Gordon equation in the limit $\kappa \rightarrow \infty$. Let us note that (28) can be written, due to (9), in the form

$$\left[\partial_0^2 - \partial_i^2 + m^2 \left(1 + \frac{m^2}{4\kappa^2} \right) \right] f = 0; \quad (29)$$

here ∂_0, ∂_i are the operators given by (9). It seems therefore that the Woronowicz operators χ_μ are better candidates for translation generators than P_μ 's. Note that the operators χ_μ already appeared in [7], [8].

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